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# Multimode difference squeezing

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**Abstract.** We define a type of multimode difference-squeezed states and show explicitly that these states are entirely embedded within the non-classical domain. We then contrast multimode difference squeezing with normal squeezing. We further analyse its delicate dependence on the modal states. All these studies emphasize the role of the concept of difference squeezing as a helpful theoretical tool for how to prepare the input modes to generate a squeezed output mode in a proper multiwave nonlinear process. Finally, we also discuss the possible connection between difference squeezing and a symmetry group.

## 1. Introduction

Information transmission is most important in communication networks. Coherent beams from laser sources are widely utilized in optical fibres to attain a high signal-to-noise ratio. However, the information precision is always bounded by the shot noise limit set intrinsically by quantum mechanics through the Heisenberg uncertainty principle. The discovery of squeezed states [1–4] has opened the way to beat the shot noise limit in a number of applications. In [5] a proposition was made to use squeezed instead of coherent light in optical systems to essentially reduce the noise in a signal. Very weak forces such as gravitational waves would also be detected by injecting a squeezed source into the unused input port of an interferometer [6]. Optical data bus technology was suggested in [7]: a squeezed state is to be applied in an optical waveguide to tap a signal-carrying waveguide and a very-low-energy-loss signal may reach many user sites without repeaters over a long distance. In principle, a noise-free signal might be achieved if it is carried by the field component which is perfectly squeezed. The quantum-mechanical nature of light is also manifested directly in higher-order squeezed states. Of a single-mode type are those introduced by Hong and Mandel [8] and by Hillery [9] (see also [10–12]). Multimode versions of higher-order squeezing were first suggested by Hillery [13] in terms of so-called sum and difference squeezing. Yet, only the simplest case of two modes was treated in [13]. The concept of Hillery's sum squeezing has been generalized to the situation of three modes by Kumar and Gupta [14] and, of an arbitrary number of modes by Nguyen and Vo [15]. Concerning the difference squeezing, Kumar and Gupta have recently considered the case of three modes [16]. In this paper we make a further generalization of [13, 16] to include any mode number. The results of [13, 16] are thus particular cases of the results we have given. In addition, some important issues (e.g. the boundaries between difference squeezed and classical states, the

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conditions imposed on the initial modal populations, the connection to symmetry groups, etc) omitted in [16] for three modes are clarified in detail here for an arbitrary number of modes.

This paper is organized as follows. In the next section we define a type of multimode difference-squeezed state and show that such states, in contrast to the sum-squeezed states [13–15], which are non-classical and have a common border with the classical states, are entirely embedded within the non-classical domain. Section 3 determines, within the short-time approximation, the conditions under which the multimode difference squeezing is related to normal squeezing. We then examine, in section 4, all the possible situations of modal states on which the system multimode difference squeezing depends. In section 5 we show that the connection, found in [13] for two modes, of the operators characterizing the multimode difference squeezing to the generators of the $su(2)$ Lie algebra is violated for a mode number greater than two. Finally, we give our conclusions.

## 2. Definition

Consider a multiwave process in a nonlinear medium in which $N$ ($\geqslant 2$) modes with frequencies $\omega_1, \omega_2, \ldots, \omega_N$ interact to create a novel mode with frequency $\Omega$ which is equal to the difference $\sum_{j=1}^{N-1} \omega_j - \omega_N$. Such an interaction process is described by the Hamiltonian ($\hbar = 1$)

$$H = \sum_{j=1}^{N} \omega_j n_j + \Omega n + f\left(b^+ a_N^+ a_{N-1} \ldots a_2 a_1 + \text{h.c.}\right) \tag{1}$$

which could be realized, say, in a system of multi-level atoms with appropriate allowed transitions among the levels. In (1) h.c. means Hermitic conjugate, $n_j = a_j^+ a_j$ ($n = b^+ b$) with $a_j^+, a_j$ ($b^+, b$) being the bosonic operators for the mode $\omega_j$ ($\Omega$) and $f$ is the effective coupling constant which is assumed to be real. The time dependence of free modes ($f = 0$) is simply proportional to $\exp(\mathrm{i}\omega_j t)$ or $\exp(\mathrm{i}\Omega t)$. The interaction among modes ($f \neq 0$) induces a slower dependence on time and we can write

$$a_j(t) = A_j(t) \exp(-\mathrm{i}\omega_j t) \qquad b(t) = B(t) \exp(-\mathrm{i}\Omega t) \tag{2}$$

where $A_j(t)$ and $B(t)$ vary slowly in time. It is convenient to work with the operators $A_j(t)$ and $B(t)$ rather than with $a_j(t)$ and $b(t)$. Let us define for modes $\omega_j$ the following 'collective' operator

$$D_\varphi(t) = \tfrac{1}{2}\left[\exp(-\mathrm{i}\varphi) A_N^+(t) \prod_{j=1}^{N-1} A_j(t) + \exp(\mathrm{i}\varphi) A_N(t) \prod_{j=1}^{N-1} A_j^+(t)\right] \tag{3}$$

where the phase $\varphi$ plays the role of the angle made by $D_\varphi$ with the real axis in the complex plane. It can be proved that for any $\varphi$

$$[D_\varphi(t), D_{\varphi+\pi/2}(t)] = \tfrac{1}{2}\mathrm{i}L(t) \tag{4}$$

with $L(t)$ given by

$$L(t) = L^+(t) = n_N(t) \prod_{j=1}^{N-1}(1 + n_j(t)) - (1 + n_N(t)) \prod_{j=1}^{N-1} n_j(t). \tag{5}$$

In view of the commutator (4) and by virtue of the Heisenberg uncertainty relation, the product of the variances of two variables $D_\varphi(t)$ and $D_{\varphi+\pi/2}(t)$ is given by

$$VD_\varphi(t)VD_{\varphi+\pi/2}(t) \geqslant \tfrac{1}{16}|\langle L(t)\rangle|^2. \tag{6}$$

In (6) $\langle\cdots\rangle = \langle\Psi|\cdots|\Psi\rangle$ with $\Psi$ the system state vector denotes the quantum average and '$V$' denotes the variance,

$$VZ \equiv \left\langle(\Delta Z)^2\right\rangle \qquad \Delta Z \equiv Z - \langle Z\rangle \tag{7}$$

for an arbitrary operator $Z$. A collective state of modes $\omega_j$ is said to be difference squeezed along a direction $\varphi$ if

$$VD_\varphi(t) - \tfrac{1}{4}|\langle L(t)\rangle| < 0. \tag{8}$$

In such squeezed states the quantum fluctuations in the variable $D_\varphi$ are reduced below their value in a symmetric minimum-uncertainty state ($VD_\varphi = VD_{\varphi+\pi/2} = \frac{1}{4}|\langle L\rangle|$) at the expense of the corresponding increased fluctuations in the variable $D_{\varphi+\pi/2}$ in order to justify the Heisenberg uncertainty relation (6). To gain a deeper insight into the above-defined multimode difference-squeezed states let us formulate $VD_\varphi(t)$ in terms of the so-called multimode quasi-probability distribution function $\mathcal{P}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ where $\alpha_j$ is a complex number. Using the Glauber–Sudarshan representation we obtain

$$VD_\varphi = \tfrac{1}{4}\langle\widetilde{L}\rangle + \int \mathcal{P}(\alpha_1, \ldots, \alpha_N)\left[\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}(\Omega t-\varphi)}\alpha_N^* \prod_{j=1}^{N-1}\alpha_j\right) - \mathrm{Re}\langle D_\varphi\rangle\right]^2 \prod_{j=1}^{N} \mathrm{d}^2\alpha_j \tag{9}$$

with

$$\widetilde{L} = n_N \prod_{j=1}^{N-1}(1+n_j) + (1-n_N)\prod_{j=1}^{N-1} n_j > 0. \tag{10}$$

Classically, the quasi-probability distribution function $\mathcal{P}$ should be definitely non-negative. Hence, a state is referred to as non-classical if the function $\mathcal{P}$ is negative. Then, in view of (9), the boundary between classical and non-classical states is determined by

$$VD_\varphi = \tfrac{1}{4}\langle\widetilde{L}\rangle. \tag{11}$$

States corresponding to $VD_\varphi < \frac{1}{4}\langle\widetilde{L}\rangle$ are obviously non-classical since for them $\mathcal{P} < 0$. Such non-classical states are not identical with the difference-squeezed states defined by (8). Because, as can be checked from (5) and (10), $\langle\widetilde{L}\rangle$ is always greater than $|\langle L\rangle|$, there exists another boundary $VD_\varphi = \frac{1}{4}|\langle L\rangle|$ which lies entirely inside the non-classical domain. This boundary separates states which are difference squeezed from states which are not difference squeezed. Evidently, the domain sandwiched between the two boundaries $VD_\varphi = \frac{1}{4}|\langle L\rangle|$ and $VD_\varphi = \frac{1}{4}\langle\widetilde{L}\rangle$ contains states which are non-classical and, at the same time, are not difference squeezed. This feature makes the difference squeezing very special compared to other known types of squeezing (e.g. normal squeezing or sum squeezing) for which non-classical and squeezed states coincide, since they have a common boundary with the classical states.

## 3. Relation between difference squeezing and normal squeezing

In the interaction process under consideration the mode with difference frequency $\Omega$ is created thanks to the nonlinear coupling between the modes with frequencies $\omega_j$. Hence, it is physically expected that difference squeezing of modes $\omega_j$ and normal squeezing of mode $\Omega$ should be related to each other. We shall establish such a relationship in this section. From the Hamiltonian (1) we derive the coupled Heisenberg equations of motion in the form

$$\dot{A}_j(t) = -\mathrm{i}f \prod_{q\neq j}^{N-1} A_q^+(t) A_N(t) B(t) \tag{12}$$

$$\dot{A}_N^+(t) = \mathrm{i}f \prod_{q=1}^{N-1} A_q^+(t) B(t) \tag{13}$$

$$\dot{B}(t) = -\mathrm{i}f \prod_{q=1}^{N} A_q(t) A_N^+(t) \tag{14}$$

where the dot denotes a time derivative. Differentiating (14) once again with respect to time and making use of (12) and (13) we obtain

$$\ddot{B}(t) = -f^2 B(t) L(t) \tag{15}$$

with $L(t)$ defined by (5). Equation (15), though simple in form, is crucial for later treatment. Its derivation, which is a bit tricky, is presented in an appendix. Since the general time-dependent solution of the set of coupled equations (12)–(14) is not available, let us approximately solve them in the short-time limit in which $B(t)$, to second order in $t$, depends on time as

$$B(t) = B(0) + t\dot{B}(0) + \tfrac{1}{2}t^2\ddot{B}(0). \tag{16}$$

Note that the system evolution over a short period of time is practically relevant because the actual interaction time is, in fact, very short in the range of picoseconds or sub-picoseconds in tiny nanosized experimented samples. Putting (14) and (15) into (16) yields

$$B(t) = B(0) - \mathrm{i}ft \prod_{q=1}^{N-1} A_q(0) A_N^+(0) - \tfrac{1}{2}f^2t^2 B(0) L(0). \tag{17}$$

It is noticeable from (17) that the time dependence shows up via $ft$ (not $t$), revealing the slow variation of the operator $B(t)$ as compared to $b(t)$ since usually $f \ll \omega_j, \Omega$. As is well known, mode $\Omega$ is said to be normally squeezed along the direction $\varphi$ if

$$VQ_\varphi(t) - \tfrac{1}{4} < 0$$

where $Q_\varphi$ is a quadrature component of mode $\Omega$

$$Q_\varphi(t) = \tfrac{1}{2}\big[B(t)\exp(-\mathrm{i}\varphi) + B^+(t)\exp(\mathrm{i}\varphi)\big]. \tag{18}$$

The explicit time dependence of $Q_\varphi(t)$ can be obtained from (17), (18) and (3)

$$Q_\varphi(t) = Q_\varphi(0) + ftD_{\varphi+\pi/2}(0) - \tfrac{1}{2}f^2t^2 C(0) Q_\varphi(0). \tag{19}$$

As for the variance $VQ_\varphi(t)$ we obtain

$$VQ_\varphi(t) = VQ_\varphi(0) + ft[\langle D_{\varphi+\pi/2}(0)Q_\varphi(0)\rangle + \langle Q_\varphi(0)D_{\varphi+\pi/2}(0)\rangle - 2\langle D_{\varphi+\pi/2}(0)\rangle\langle Q_\varphi(0)\rangle] + f^2t^2\big[VD_{\varphi+\pi/2}(0) + \langle Q_\varphi(0)\rangle\langle L(0)Q_\varphi(0)\rangle - \tfrac{1}{2}\langle Q_\varphi(0)L(0)Q_\varphi(0)\rangle - \tfrac{1}{2}\langle L(0)Q_\varphi^2(0)\rangle\big]. \tag{20}$$

Anticipating no correlations between modes $\omega_j$ and mode $\Omega$ at $t = 0$ reduces (20) to

$$VQ_\varphi(t) = VQ_\varphi(0) + f^2t^2[VD_{\varphi+\pi/2}(0) - \langle L(0)\rangle VQ_\varphi(0)]. \tag{21}$$

Furthermore, if before the interaction process takes place there are no photons in mode $\Omega$ or this mode is in a coherent state, i.e. $VQ_\varphi(0) = \frac{1}{4}$, then we can cast (21) into the form

$$VQ_{\varphi-\pi/2}(t) - \tfrac{1}{4} = f^2t^2\big[VD_\varphi(0) - \tfrac{1}{4}\mathrm{sign}(\langle L(0)\rangle)|\langle L(0)\rangle|\big]. \tag{22}$$

Since the definition (5) does not guarantee the positivity of $\langle L(0)\rangle$, the consequences to be gained from (22) depend on the sign of $\langle L(0)\rangle$. The latter is determined by the initial mode populations. It follows from (5) that

$$\langle L(0)\rangle > 0 \qquad \text{when} \quad \langle n_N(0)\rangle > \mathcal{N}_N \tag{23}$$

and

$$\langle L(0)\rangle \leqslant 0 \qquad \text{when} \quad \langle n_N(0)\rangle \leqslant \mathcal{N}_N \tag{24}$$

where

$$\mathcal{N}_N = \frac{\prod_{j=1}^{N-1}\langle n_j(0)\rangle}{\prod_{j=1}^{N-1}(1 + \langle n_j(0)\rangle) - \prod_{j=1}^{N-1}\langle n_j(0)\rangle}. \tag{25}$$

When $\langle n_N(0)\rangle \leqslant \mathcal{N}_N$, mode $\Omega$ departs from its initial state with an increasing variance. It therefore cannot evolve into a squeezed state whatever the initial state of modes $\omega_j$. On the other hand, when $\langle n_N(0)\rangle > \mathcal{N}_N$, equation (22) expresses an interesting relationship between normal squeezing of mode $\Omega$ and difference squeezing of modes $\omega_j$. Namely, if at $t = 0$ modes $\omega_j$ are not difference squeezed then at $t > 0$ mode $\Omega$ will not be normally squeezed either. On the other hand, if at $t = 0$ modes $\omega_j$ are difference squeezed along some direction $\varphi$, then this makes mode $\Omega$ at an immediate later time normally squeezed along the direction $\varphi - \pi/2$. That is, the directions of difference squeezing and normal squeezing are perpendicular to each other. Compared to the case of sum squeezing [15], difference squeezing versus normal squeezing, i.e. the relationship (22), requires an additional constraint to be imposed on the mode populations, equation (23). In particular, for the two-mode case this constraint implies that the lower-frequency mode must be populated more than the higher-frequency mode: $\langle n_2(0)\rangle > \langle n_1(0)\rangle$ (for $\omega_2 < \omega_1$), in agreement with [13]. This point was not argued at all in [16] when dealing with three modes. Following (23), the required inequality for the population of the modes for $N = 3$ is

$$\langle n_3(0)\rangle > \frac{\langle n_1(0)\rangle\langle n_2(0)\rangle}{1 + \langle n_1(0)\rangle + \langle n_2(0)\rangle}. \tag{26}$$

## 4. Difference squeezing as governed by modal states

By definition, the difference squeezing bears a collective character. Naturally, this type of squeezing depends on the state of individual modes that constitute the multimode state. In this section we shall study such delicate dependences in detail. To that end, it is convenient to work with direction-independent conditions for the occurrence of squeezing. It is known from [13] that a mode $\omega_k$ is normally squeezed if and only if

$$\tfrac{1}{2} > |\langle A_k\rangle - \langle A_k\rangle^2| - \langle n_k\rangle + |\langle A_k\rangle|^2 > 0. \tag{27}$$

Next, as learnt from [15], if $K$ uncorrelated modes $\omega_1, \omega_2, \ldots, \omega_K$ are sum squeezed, then

$$\left|\prod_{k=1}^{K}\langle A_k^2\rangle - \prod_{k=1}^{K}\langle A_k\rangle^2\right| > \prod_{k=1}^{K}\langle n_k\rangle - \prod_{k=1}^{K}|\langle A_k\rangle|^2. \tag{28}$$

For a later purpose, it is necessary to determine the upper bound of the left-hand side of (28). By virtue of the Schwarz inequality and the fact that $(x + y/2)^2 \geqslant x(x + y)$ for any $x, y > 0$, we are able to derive the inequality

$$\prod_{k=1}^{K}\langle n_k\rangle - \prod_{k=1}^{K}|\langle A_k\rangle|^2 + \tfrac{1}{2}\left(\prod_{k=1}^{K}\langle 1+n_k\rangle - \prod_{k=1}^{K} n_k\right) > \left|\prod_{k=1}^{K}\langle A_k^2\rangle - \prod_{k=1}^{K}\langle A_k\rangle^2\right|. \tag{29}$$

Combining (28) and (29) yields the necessary and sufficient condition for the $K$ uncorrelated modes to be sum squeezed as follows:

$$\tfrac{1}{2}\left(\prod_{k=1}^{K}\langle 1+n_k\rangle - \prod_{k=1}^{K} n_k\right) > \left|\prod_{k=1}^{K}\langle A_k^2\rangle - \prod_{k=1}^{K}\langle A_k\rangle^2\right| - \prod_{k=1}^{K}\langle n_k\rangle + \prod_{k=1}^{K}|\langle A_k\rangle|^2 > 0. \tag{30}$$

Concerning the multimode difference squeezing, it is easy to verify the following condition: a set of $N$ modes are difference squeezed if and only if

$$\left|\left\langle\left(A_N^+\prod_{j=1}^{N-1}A_j\right)^2\right\rangle - \left\langle A_N^+\prod_{j=1}^{N-1}A_j\right\rangle^2\right| > \left\langle(1+n_N)\prod_{j=1}^{N-1}n_j\right\rangle - \left|\left\langle A_N^+\prod_{j=1}^{N-1}A_j\right\rangle\right|^2$$

which for uncorrelated modes becomes

$$\left|\langle A_N^{+2}\rangle\prod_{j=1}^{N-1}\langle A_j^2\rangle - \langle A_N^+\rangle^2\prod_{j=1}^{N-1}\langle A_j\rangle^2\right| > (1+\langle n_N\rangle)\prod_{j=1}^{N-1}\langle n_j\rangle - |\langle A_N^+\rangle|^2\prod_{j=1}^{N-1}|\langle A_j\rangle|^2. \tag{31}$$

Now we are in the position to study difference squeezing versus modal states. We have examined all the possible situations and the results obtained are summarized in the form of a set of theorems as follows.

**Theorem 1.** *If there is at least one mode which is in a Fock state, then the multimode system is not difference squeezed.*

**Proof.** Being in a Fock state of mode $\omega_l$ means

$$\langle A_l^q\rangle = \langle A_l\rangle^q = \langle A_l^{+q}\rangle = \langle A_l^+\rangle^q = 0 \qquad q = 1, 2, \ldots \quad \text{and} \quad \langle n_l\rangle > 0. \tag{32}$$

The properties (32) make the left-hand side of (31) vanish identically, while its right-hand side is equal to $(1 + \langle n_N\rangle)\prod_{j=1}^{N-1}\langle n_j\rangle > 0$. The inequality (31) does not hold and, thus, difference squeezing cannot appear. □

**Theorem 2.** *If all modes are coherent, then the multimode system is not difference squeezed.*

**Proof.** This theorem is trivially true since a coherent state of mode $\omega_l$ satisfies the properties

$$\left\langle A_l^q\right\rangle^* = \langle A_l\rangle^{*q} = \left\langle A_l^{+q}\right\rangle = \left\langle A_l^+\right\rangle^q \neq 0 \qquad \text{and} \qquad \langle n_l\rangle = |\langle A_l\rangle|^2 = \left|\left\langle A_l^+\right\rangle\right|^2 > 0. \tag{33}$$

Applying (33) to (31) leads to left-hand side $= 0$ and right-hand side $= \langle n_N\rangle \prod_{j=1}^{N-1}\langle n_j\rangle > 0$ which implies the absence of difference squeezing. □

**Theorem 3.** *If mode $\omega_N$ is squeezed but all the other modes are coherent, then the multimode system is not difference squeezed.*

**Proof.** Taking (31) and (33) into account, difference squeezing would arise if

$$\left|\left\langle A_N^{+2}\right\rangle - \left\langle A_N^+\right\rangle^2\right| > 1 + \langle n_N\rangle - \left|\left\langle A_N^+\right\rangle\right|^2. \tag{34}$$

Since mode $\omega_N$ is squeezed the inequalities like (27) should hold

$$\tfrac{1}{2} > \left|\left\langle A_N^2\right\rangle - \langle A_N\rangle^2\right| - \langle n_N\rangle + |\langle A_N\rangle|^2 > 0. \tag{35}$$

Surely, equation (34) cannot be fulfilled because of (35). Hence, there is no difference squeezing. □

**Theorem 4.** *If there is a mode $\omega_k$ with $1 \leqslant k \leqslant N-1$ which is being squeezed and all the remaining modes are coherent, then the multimode system will be difference squeezed when $\langle n_k\rangle < \langle n_N\rangle/2$.*

**Proof.** Upon using (33) the condition (31) for difference squeezing simplifies to

$$\left|\left\langle A_k^2\right\rangle - \langle A_k\rangle^2\right| > \frac{\langle n_k\rangle}{\langle n_N\rangle} + \langle n_k\rangle - |\langle A_k\rangle|^2. \tag{36}$$

Since mode $\omega_k$ is squeezed the inequalities (27) should hold. Comparing (27) and (36) reveals that both of them will be satisfied when

$$\frac{\langle n_k\rangle}{\langle n_N\rangle} < \tfrac{1}{2}. \tag{37}$$

Whenever (37) is violated, difference squeezing is absent. It is worth noting here that the populations of modes other than $\omega_k$ and $\omega_N$ play no roles in generating the difference squeezing. □

**Theorem 5.** *If there are $Q$ modes $\omega_1, \omega_2, \ldots, \omega_Q$ with $1 < Q \leqslant N-1$ which is squeezed and all the other modes are coherent, then the multimode system will be difference squeezed when the modes $\omega_1, \omega_2, \ldots, \omega_Q$ are sum squeezed and, in addition to that, their populations must satisfy the constraint*

$$\prod_{q=1}^{Q}\langle n_q\rangle \bigg/ \left(\prod_{q=1}^{Q}\langle 1 + n_q\rangle - \prod_{q=1}^{Q}\langle n_q\rangle\right) < \langle n_N\rangle/2.$$

**Proof.** In this situation the condition (31) reads

$$\left|\prod_{q=1}^{Q}\langle A_q^2\rangle - \prod_{q=1}^{Q}\langle A_q\rangle^2\right| > \frac{\prod_{q=1}^{Q}\langle n_q\rangle}{\langle n_N\rangle} + \prod_{q=1}^{Q}\langle n_q\rangle - \prod_{q=1}^{Q}|\langle A_q\rangle|^2. \tag{38}$$

Obviously, if the $Q$ modes $\{\omega_q\}$ are not sum squeezed, then (38) is violated due to (28). Yet, sum squeezing is not sufficient. Paying attention to (30), the occurrence of difference squeezing requires, in addition, that the populations of modes $\{\omega_q\}$ and mode $\omega_N$ satisfy the inequality

$$\frac{\prod_{q=1}^{Q}\langle n_q\rangle}{\langle n_N\rangle} < \frac{1}{2}\left(\prod_{q=1}^{Q}\langle 1+n_q\rangle - \prod_{q=1}^{Q}\langle n_q\rangle\right) \tag{39}$$

with no constraints on the population of the remaining modes, i.e. modes $\omega_{Q+1}, \omega_{Q+2}, \ldots, \omega_{N-1}$. This theorem contains the preceding one as a particular case. □

**Theorem 6.** *If modes* $\omega_1, \omega_2, \ldots, \omega_Q$ *with* $1 \leqslant Q \leqslant N-1$ *as well as mode* $\omega_N$ *are squeezed while the remaining modes are coherent, then the multimode system may or may not be difference squeezed.*

**Proof.** In this case, a definite determination of the constraints for the system to be or not to be difference squeezed is impossible in general. We shall thus proceed by means of explicit examples in certain limits. Let us examine the inequality

$$|X| > Y \tag{40}$$

where we have identified

$$X \equiv \langle A_N^{+2}\rangle \prod_{q=1}^{Q}\langle A_q^2\rangle - \langle A_N^+\rangle^2 \prod_{q=1}^{Q}\langle A_q\rangle^2 \tag{41}$$

and

$$Y \equiv (1+\langle n_N\rangle)\prod_{q=1}^{Q}\langle n_q\rangle - \left|\langle A_N^+\rangle\right|^2 \prod_{q=1}^{Q}\left|\langle A_q\rangle\right|^2. \tag{42}$$

The multimode system state vector $|\Psi\rangle$ underlying this theorem reads

$$|\Psi\rangle = \prod_{q=1}^{Q}\otimes|\alpha_q, z_q\rangle \otimes \prod_{j=Q+1}^{N-1}\otimes|\alpha_j, 0\rangle \otimes |\alpha_N, z_N\rangle \tag{43}$$

where

$$|\alpha_k, z_k\rangle = \exp\left(z_k^* A_k^2 - z_k A_k^{+2}\right)\exp\left(\alpha_k A_k^+ - \alpha_k^* A_k\right)|0\rangle \tag{44}$$

with

$$z_k = \rho_k \exp(\mathrm{i}\vartheta_k) \qquad \alpha_k = r_k \exp(\mathrm{i}\theta_k) \qquad \text{and} \qquad \rho_k, \vartheta_k, \alpha_k, r_k \quad \text{real} \tag{45}$$

describes a squeezed state ($k = 1, 2, \ldots, Q, N$) or a coherent state ($Q < k < N$). The quantum averages of interest are

$$\langle A_k\rangle \equiv \langle\Psi|A_k|\Psi\rangle = \alpha_k = \langle A_k^+\rangle^* \tag{46}$$

$$\langle A_k^2\rangle \equiv \langle\Psi|A_k^2|\Psi\rangle = \alpha_k^2 - \mathrm{e}^{\mathrm{i}\vartheta_k}\cosh\rho_k\sinh\rho_k = \langle A_k^{+2}\rangle^* \tag{47}$$

$$\langle n_k\rangle \equiv \langle\Psi|n_k|\Psi\rangle = |\alpha_k|^2 + \sinh^2\rho_k. \tag{48}$$

We first consider a limiting case in which $r_1 = \cdots = r_Q = r_N = r = \mathcal{O}(1)$ and $\rho_1 = \cdots = \rho_Q = \rho_N = \rho \ll 1$. Then, we can expand in powers in $\rho$ and, as a result, we have

$$|X| = \rho r^{2Q}\left(\mathrm{e}^{-\mathrm{i}\vartheta_N + 2\mathrm{i}\sum_{q=1}^{Q}\theta_q} + \mathrm{e}^{-2\mathrm{i}\theta_N}\sum_{q=1}^{Q}\mathrm{e}^{\mathrm{i}(\theta_q + 2\sum_{q\neq q}^{Q}\theta_p)}\right) + \mathcal{O}(\rho^2) \tag{49}$$

and

$$Y = r^{2Q} + \mathcal{O}(\rho^2). \tag{50}$$

Clearly, from (49) and (50), for arbitrary phases, to leading order in $\rho$, $|X| \leqslant \mathcal{O}(\rho) < Y = \mathcal{O}(1)$ indicating no difference squeezing of the multimode system. Another limit we next consider is $r_N = R = \mathcal{O}(1)$ and $r_1 = \cdots = r_Q = \rho_1 = \cdots = \rho_Q = \rho_N = \rho \ll 1$. In this limit,

$$|X| = \rho^Q R^2 \exp\left[\mathrm{i}\sum_{q=1}^{Q}\vartheta_q - 2\mathrm{i}\theta_N\right] + \mathcal{O}(\rho^{Q+1}) \tag{51}$$

and

$$Y = \rho^{2Q}\left[2^Q + \left(2^Q - 1\right)R^2\right] + \mathcal{O}(\rho^{2Q+2}). \tag{52}$$

Clearly, from (51) and (52), for arbitrary phases, to leading order in $\rho$, $|X| = \mathcal{O}(\rho^Q) > Y = \mathcal{O}(\rho^{2Q})$, indicating the occurrence of difference squeezing. Two different limits yield two opposite results. This has proven the theorem. Note that in the above-considered limits the relative ratio between the coherence and squeezing amplitudes are of importance but no role is played by the phases. In general, all the modal parameters, including phases, influence the possibility of the system difference squeezing. □

## 5. Relation between difference squeezing and a symmetry group

The relation between various types of squeezing and Lie algebras has been exploited by a number of authors (see, e.g., [17–19]). It was demonstrated in [13] that for two modes the sum-squeezing characteristic operators form a representation of the $su(1, 1)$ Lie algebra, whereas the difference-squeezing characteristic operators form a representation of the $su(2)$ Lie algebra. The connection of sum squeezing to the $su(1, 1)$ symmetry group has been shown (see [15]) to hold in the most general case, i.e. for an arbitrary mode number $N \geqslant 2$. To test whether there is a symmetry group to which the difference squeezing is related for an arbitrary $N \geqslant 2$, we calculate the general commutator $[D_\varphi, L]$. As a result, we arrive at

$$[D_\varphi, L] = \tfrac{1}{2}\left[\exp(-\mathrm{i}\varphi)M_1 A_N^+ \prod_{j=1}^{N-1} A_j + \exp(\mathrm{i}\varphi)M_2 A_N \prod_{j=1}^{N-1} A_j^+\right] \tag{53}$$

where

$$M_1 = \left[ n_N \prod_{j=1}^{N-1}(n_j+1) - 2(n_N+1)\prod_{j=1}^{N-1} n_j + (n_N+2)\prod_{j=1}^{N-1}(n_j-1) \right] \tag{54}$$

and

$$M_2 = \left[ 2n_N \prod_{j=1}^{N-1}(n_j+1) - (n_N+1)\prod_{j=1}^{N-1} n_j - (n_N-1)\prod_{j=1}^{N-1}(n_j+2) \right]. \tag{55}$$

As can be checked from (53) and (55), only for $N = 2$, in which case $M_1 = -M_2 = -2$, the three operators $D_\varphi$, $D_{\varphi+\pi/2}$ and $L$ form a closed group: they are proportional to the generators of the $su(2)$ Lie group, as was pointed out in [13]. Nevertheless, for $N \geqslant 3$ the operators $\{D_\varphi, D_{\varphi+\pi/2}, L\}$ do not form an algebra and, thus, there is no connection of the difference squeezing to any symmetry group. This point was completely skipped in [16] when dealing with $N = 3$.

## 6. Conclusion

In conclusion, we have defined a type of multimode difference squeezing which is relevant to physical processes in which a difference-frequency is generated in a nonlinear medium via multiwave coupling. Our consideration holds for arbitrary $N \geqslant 2$ modes and is therefore a natural generalization of the cases studied previously by Hillery [13] for $N = 2$ and, by Kumar and Gupta [16] for $N = 3$. The defined difference squeezing is shown to be very special compared to other known types of squeezing, in the sense that around its existence domain there are no nearby classical states at all. Being a collective multimode state, the difference squeezing is delicately governed by the states of individual modes. Our detailed analysis reveals that (a) the system can never be difference squeezed if among the modes there is at least one mode which is in a Fock state or all the modes are coherent or mode $\omega_N$ is squeezed but all the remaining ones are coherent and (b) the system may or may not be difference squeezed in all other situations for the modal states. In short, squeezing of at least one among the modes $\omega_1, \omega_2, \ldots, \omega_{N-1}$ is necessary (but not sufficient) for $N$-mode difference squeezing. In particular, difference squeezing of the system is not guaranteed even when all the $N$ modes are individually squeezed (see theorem 6 with $Q = N - 1$). We have also proven that whenever the modal initial populations satisfy a certain constraint the difference-frequency generation converts difference squeezing to normal squeezing with their squeezing directions being perpendicular to each other. Finally, we have found explicitly that the relation between the difference squeezing and a symmetry group exists only in the two-mode regime, in which case the symmetry group is the $su(2)$ Lie one. Concerning the physics one might ask how to measure the quantity associated with the 'collective' operator $D_\varphi$ defined by (3)? In fact, it is not necessary to measure it directly. This operator $D_\varphi$ is introduced just to show that one is able to produce the output mode $\Omega$ in a squeezed state if beforehand one prepared the input modes $\{\omega_j\}$ in a difference-squeezed state (see the relation (22)). The question of how to prepare a difference-squeezed state of the input modes is guided by the theorems proven in section 4. Namely, if the input modes are those underlying the situations of theorems 1–3, then difference squeezing is impossible and thus squeezed output cannot be generated. Under the situations of theorems 4–6 certain conditions for the modal populations should be met to obtain the output mode as a squeezed one. In other words, the concept of difference squeezing can be looked upon as a useful intermediate theoretical tool to help find out how to prepare the input modes to generate the squeezed output mode through a nonlinear multimode system. Finally, from a general point of view, the problem under consideration could be of fundamental interest because it once again directly demonstrates a non-classical effect originated from the quantum nature of light. The concept of difference squeezing is, however, by no means limited to photons. It might apply equally to elementary excitations such as excitons, biexcitons, phonons, plasmons, etc in condensed matter at low densities since these behave, to a good approximation, like ideal bosons.

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## Appendix

This appendix derives equation (15). From (14) it follows that

$$\begin{aligned}\ddot{B} &= -\mathrm{i}f\left(\prod_{k=1}^{N-1} A_k A_N^+\right)^{\cdot} \\ &= -\mathrm{i}f\left(\dot{A}_1 A_2 \ldots A_{N-1} A_N^+ + A_1 \dot{A}_2 A_3 \ldots A_{N-1} A_N^+ + \cdots + A_1 \ldots A_{N-1} \dot{A}_N^+\right).\end{aligned} \tag{A1}$$

Using (12) and (13) in (A1) yields

$$\begin{aligned}\ddot{B} = -f^2 B[&n_2 \ldots n_{N-1}(1+n_N) + (1+n_1)n_3 \ldots n_{N-1}(1+n_N) \\ &+ \cdots + (1+n_1)\ldots(1+n_{N-2})(1+n_N) - (1+n_1)\ldots(1+n_{N-1}) \\ &+ n_N(1+n_1)\ldots(1+n_{N-1}) - n_N(1+n_1)\ldots(1+n_{N-1})].\end{aligned} \tag{A2}$$

On the last line of (A2) we have used a little trick. Namely, we have added and subtracted by hand the same quantity $n_N(1+n_1)\ldots(1+n_{N-1})$. Leaving the term preceding the last term as it is and grouping the two terms with a minus sign within the square brackets cast (A2) into

$$\begin{aligned}\ddot{B} = -f^2 B\Bigg\{&(1+n_N)[n_2 \ldots n_{N-1} + (1+n_1)n_3 \ldots n_{N-1} \\ &+ \cdots + (1+n_1)\ldots(1+n_{N-3})n_{N-2} + (1+n_1)\ldots(1+n_{N-2}) \\ &-(1+n_1)\ldots(1+n_{N-1})] + n_N \prod_{k=1}^{N-1}(1+n_k)\Bigg\}.\end{aligned} \tag{A3}$$

Within the square brackets of (A3) we sum up successively term by term from bottom to top and obtain

$$\ddot{B} = -f^2 B\left\{\left[-(1+n_N)\prod_{k=1}^{N-1} n_k\right] + n_N \prod_{k=1}^{N-1}(1+n_k)\right\}. \tag{A4}$$

Equation (A4), due to (5), is nothing else but the desired equation (15).

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